

On the ratio ergodic theorem for group actions

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Abstract

We study the ratio ergodic theorem (RET) of Hopf for group actions. Under a certain technical condition, if a sequence of sets $\{F_n\}$ in a group satisfy the RET, then there is a finite set E such that $\{EF_n\}$ satisfies the Besicovitch covering property. Consequently for the abelian group $G = \oplus_{n=1}^{\infty} \mathbb{Z}$ there is no sequence $F_n \subseteq G$ along which the RET holds, and in many finitely generated groups, including the discrete Heisenberg group and the free group on ≥ 2 generators, there is no (sub)sequence of balls, in the standard generators, along which the RET holds.

On the other hand, in groups with polynomial growth (including the Heisenberg group, to which our negative results apply) there always exists a sequence of balls along which the RET holds if convergence is understood as a.e. convergence in density (i.e. omitting a sequence of density zero).

1 Introduction

Let G be a countable group acting from the left by measure-preserving transformations on a measure space (X, \mathcal{B}, μ) , with the action of $g \in G$ on $x \in X$ written $x \mapsto T^g x$. We assume the action is ergodic. For a finite set $F \subseteq G$ and $\varphi : X \rightarrow \mathbb{R}$ let

$$S_F(\varphi) = \sum_{g \in F} \varphi \circ T^g$$

The asymptotic behavior of $S_{F_n}(\varphi)$ as F_n exhausts the group, in some sense, is the subject of the ergodic theorem. In this paper we are interested in the situation when μ is an infinite (without loss of generality σ -finite) measure, in which case the appropriate quantity to consider are the ratios

$$R_F(\varphi, \psi) = \frac{S_F(\varphi)}{S_F(\psi)}$$

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One says that G satisfies a ratio ergodic theorem along $\{F_n\}$ if for every ergodic measure-preserving action of G on a non-atomic measure space, and every $\varphi, \psi \in L^1(\mu)$ with $\int \psi d\mu \neq 0$, we have

$$R_n(\varphi, \psi) \rightarrow \frac{\int \varphi d\mu}{\int \psi d\mu} \quad \mu\text{-a.e.}$$

For $G = \mathbb{Z}$ and $F_n = [1, n] \cap \mathbb{Z}$ the ratio ergodic theorem was proved by Hopf in 1937, only a few years after the ergodic theorems of von Neumann and Birkhoff. Unlike the latter theorems, however, which have been extended to very general classes of groups (see [10] for a recent survey), extensions of Hopf's theorem have been slow to appear. Part of the reason is that, for a time, it was believed that no extension is possible, due to an example of Brunel and Krengel [8], who showed for \mathbb{Z}^d , $d \geq 2$, that ratio ergodic theorem fails along $F_n = [0, n]^d$. Nevertheless there is a ratio ergodic theorem for actions of \mathbb{Z}^d along symmetric cubes $F_n = [-n, n]^d \cap \mathbb{Z}^d$. This was first proved by Feldman under additional assumptions on the dynamics of the action [4], and we proved the general case in [7] (also for some more general sequences F_n).¹

The prospects for groups other than \mathbb{Z}^d has remained unclear. A pertinent fact from [7] is that a certain maximal inequality, which is central to most existing proofs, is actually *equivalent*, in the present context, to $\{F_n\}$ satisfying the (right) Besicovitch covering property (Definition 2.1 below). Sufficiency of this property was observed earlier by Becker [2] (see also [11, 9]). The Besicovitch property is quite rare, and its failure puts into question the validity of the ratio ergodic theorem for many groups. But, on the other hand, as far as we know the maximal inequality and the ratio ergodic theorem are not equivalent. Indeed for \mathbb{Z}^d it took several decades to prove the latter once the former became available.

In this paper we present two main results. The first shows that, indeed, the ratio ergodic theorem is quite rare, and is closely linked to the Besicovitch property. The second, on the other hand, shows that a certain weakening of it does hold more generally, including in groups where the strong version above fails.

We begin with main negative result, which requires the following definition. Let us say that $\{F_n\}$ is almost central if for every $g \in G$ there is a finite set E with $F_n g \subseteq E F_n$ for all n (here and throughout we write $AB = \{ab : a \in A, b \in B\}$, etc.). This holds trivially in abelian groups, and also for balls in any finitely generated group.

Theorem 1.1. *If $\{F_n\}$ is almost central and satisfies the ratio ergodic theorem, then there is a finite set E such that $\{E F_n\}$ is Besicovitch.*

¹Bowen and Nevo have also recently obtained a variant of the ratio ergodic theorem for free groups, but with some additional randomization which makes the problem somewhat different.

For a marginally stronger statement see Theorem 2.7. It seems possible that the Besicovitch property is necessary in general for a ratio ergodic theorem, but this remains open.

We give two main applications. First,

Theorem 1.2. *Let $\mathbb{Z}^\infty = \oplus_{n=1}^\infty \mathbb{Z}$. Then the ratio ergodic theorem fails along every sequence $F_n \subseteq \mathbb{Z}^\infty$.*

The group \mathbb{Z}^∞ is extremely “nice” – it is abelian, amenable and residually finite. As such it is quite surprising that the ratio ergodic theorem should fail. However one might object that it is “infinite-dimensional”. So suppose now that G is generated by a finite symmetric set A and let $B_n = A^n$ be the associated “balls”.

Theorem 1.3. *Let G be finitely generated and $\{B_n\}$ the sequence of balls with respect to some finite generating set. Suppose that no sequence of balls has the Besicovitch property. Then the ratio ergodic theorem fails along every sub-sequence $F_i = B_{n(i)}$.*

In particular, in the discrete Heisenberg group with the standard generator set there is no sequence of balls which satisfies the ratio ergodic theorem (see Section 2.3). The same holds for free groups on ≥ 2 generators. In these examples we cannot yet rule out the existence of some other sequence along which it does hold, and one should note that there are finitely generated groups, such as the lamplighter groups, for which balls are not the right averaging sets to consider. But for groups of polynomial growth all known ergodic theorems do hold along balls, and it would be quite surprising if some other good sequence exists.

We turn now to our positive result which shows that, if one accepts a slightly weaker notion of convergence, then there is a version of the ratio ergodic theorem which holds in greater generality. Recall that a group G has polynomial growth if the balls B_n satisfy $|B_n| \leq c'n^c$ for constants c, c' . Define the (upper) density of a set $I \subseteq \mathbb{N}$ of integers by

$$\bar{d}(I) = \limsup_{N \rightarrow \infty} \frac{1}{N} |I \cap [1, N]|$$

A sequence a_n converges to a in density if

$$\bar{d}(n : |a - a_n| > \varepsilon) = 0 \quad \text{for every } \varepsilon > 0$$

We denote this limit by $a_n \xrightarrow{\bar{d}} a$ or $\bar{d}\text{-}\lim a_n = a$. This operator satisfies all the usual properties of limits.

Theorem 1.4. *Let G be a group of polynomial growth and B_n as above. Then there*

is a subsequence $F_i = B_{n(i)}$ along which the ratio ergodic theorem holds in density, i.e.

$$R_{F_i}(\varphi, \psi) \xrightarrow{\bar{d}} \frac{\int \varphi d\mu}{\int \psi d\mu} \quad \mu\text{-a.e.} \quad (1)$$

for any ergodic measure-preserving action of G and any $\varphi, \psi \in L^1(\mu)$ with $\int \psi d\mu \neq 0$.

The proof is given in Section 3. Thus there are cases, such as the discrete Heisenberg group, where no sequence of balls satisfied the ratio ergodic theorem, but there exist sequences along for which the density version is valid. Our arguments are special to groups of polynomial growth but some parts carry over also to groups of sub-exponential growth (groups with $|B_n| = o(c^n)$ for all $c > 1$). We do not know if and when similar modifications of the ratio ergodic theorem hold in more general groups.

There are other possible weakenings of the ratio ergodic theorem. One possibility is to require a.e. pointwise convergence of the ratios to other limit functions. This phenomenon has been recently observed in certain algebraic settings involving “large” groups acting on infinite measure spaces, see e.g. the introduction of [3].² However, our negative results exclude this as well; in the proofs we construct actions for which the ratios diverge.

The rest of the paper is divided into two sections: In Section 2 we develop the necessary combinatorics and prove Theorems 1.3, 1.2 and 1.1. We prove Theorem 1.4 in Section 3.

2 Besicovitch is necessary

2.1 Combinatorial preliminaries

In this section $\{F_n\}$ denotes a sequence of finite subsets of G , all containing the identity element 1_G . We begin with some combinatorial definitions.

A collection $\{E_i\}_{i \in I}$ of subset of G is said to have multiplicity k at a point g if g belongs to k of the sets. The multiplicity of $\{E_i\}$ is the smallest k such that all points have multiplicity $\leq k$. The following definition is classical in analysis where, instead of translates of sets in a group, one considers balls in a metric space.

Definition 2.1. $\{F_n\}$ satisfies the Besicovitch covering property (or, more concisely, $\{F_n\}$ is Besicovitch) if there is a constant C such that, for every finite $A \subseteq G$ and any family of sets of the form $\{F_{n(g)}g\}_{g \in A}$ there is a subset $A' \subseteq A$ such that $\{F_{n(g)}g\}_{g \in A'}$

²I am grateful to Amos Nevo for drawing my attention to this phenomenon.

covers A and has multiplicity $\leq C$; equivalently,

$$1_A \leq \sum_{g \in A'} 1_{F_{n(g)}g} \leq C$$

It is easy to see that any finite sequence $\{F_n\}_{n=1}^N$ is Besicovitch and that $\{F_n\}_{n=1}^\infty$ is Besicovitch if and only if $\{F_n\}_{n=n_0}^\infty$ is Besicovitch for every n_0 . In this section we rely primarily on the following characterization of the Besicovitch property. Define a (right) incremental sequence to be a finite sequence $(F_{n(i)}g_i)_{i=1}^k$ such that $g_j \notin \bigcup_{i < j} F_{n(i)}g_i$ and $n(1) \geq n(2) \geq \dots \geq n(k)$.

Proposition 2.2. *If $\{F_n\}$ is not Besicovitch then for every k there is an incremental sequence of multiplicity k (equivalently, with 1_G belonging k members of the sequence). If in addition F_n are symmetric and increasing, the converse holds.*

Proof. We include the standard proof for completeness. If $\{F_n\}$ is not Besicovitch then, given k , there is a family $\{F_{n(i)}g_i\}_{i=1}^\ell$ such that any sub-collection covering all the g_i is of multiplicity k . We may assume $n(i)$ are non-increasing. Choose an incremental subsequence $\{F_{n(i_m)}g_{i_m}\}$ greedily: let $i_1 = 1$, and if i_1, \dots, i_m are defined take i_{m+1} to be the minimal $i > i_m$ satisfying $g_i \notin \bigcup_{j \leq m} F_{n(i_j)}g_{i_j}$. The resulting sequence covers all the g_i (here we use $1_G \in F_n$), thus has multiplicity k . We can assume the multiplicity is realized at 1_G by applying an appropriate right translation to the sets.

In the other direction, given k let $\{F_{n(i)}g_i\}$ be an incremental sequence of multiplicity k . Thus $g_i \notin F_{n(j)}g_j$ for all $i > j$; by symmetry $g_j \notin F_{n(j)}^{-1}g_i = F_{n(j)}g_i$. Since $n(j) \geq n(i)$ this shows that $g_j \notin F_{n(i)}g_i$ also for $i < j$. Thus this holds for all $i \neq j$, and the only sub-collection of $\{F_{n(i)}g_i\}$ that covers all the g_i is the full sequence, whose multiplicity is k . Since k was arbitrary, $\{F_n\}$ is not Besicovitch. \square

Our main interest is in sequences for which the Besicovitch property fails. We require a slightly stronger property:

Definition 2.3. $\{F_n\}$ is strongly non-Besicovitch if for every finite $E \subseteq G$ there is a finite set $E \subseteq \tilde{E} \subseteq G$ with $1_G \in \tilde{E}$ such that $\{\tilde{E}F_n\}$ is not Besicovitch.

We note two situations where this property holds: first, when no sequence $F'_n \supseteq F_n$ is Besicovitch (take $\tilde{E} = E \cup \{1_G\}$). Second, if G is finitely generated, B_n are balls, and no sub-sequence of balls $\{B_{n(i)}\}_{i=1}^\infty$ is Besicovitch then every sub-sequence is strongly non-Besicovitch; indeed given a finite set E take $\tilde{E} = B_m$, so that $\tilde{E}B_n = B_{n+m}$.

The sets we consider later will also satisfy the following property, which was already mentioned in the introduction:

Definition 2.4. $\{F_n\}$ is almost central if for every $g \in G$ there is a finite set $E \subseteq G$ with $F_n g \subseteq EF_n$ for all n .

The two primary examples are when G is abelian, in which case we can take $E = \{g\}$; and when $F_n = B_{k(n)}$ are balls, since then if $g \in B_m$ then $F_n g \subseteq B_m F_n$. The latter example shows that this property does not actually have any connection to abelianness of the group, since it holds for balls in any finitely generated group. It is clear that the definition is equivalent to the statement that for every finite $E \subseteq G$ there is a finite $E' \subseteq G$ with $F_n E \subseteq E' F_n$ for all n .

We now derive some properties of strongly non-Besicovitch and almost-central sequences.

Lemma 2.5. *Suppose that $\{F_n\}$ is strongly non-Besicovitch. For every finite $E \subseteq G$ there is a finite $\tilde{E} \subseteq G$ containing E , and such that for every finite $H \subseteq G$ there are arbitrarily long incremental sequences $\{F_{n(i)} g_i\}_{i=1}^\ell$ satisfying*

- (a) $Eg_i \cap F_{n(j)} g_j = \emptyset$ and $Eg_i \cap Eg_j = \emptyset$ for $i > j$.
- (b) $Eg_i \cap H = \emptyset$ for all i .
- (c) $\tilde{E}^{-1} \cap F_{n(i)} g_i \neq \emptyset$ for all i .

Proof. Let E be given and assume without loss of generality that $1_G \in E$. Let \tilde{E} be associated to $E^{-1}E$ as in Definition 2.3. Let $H \subseteq G$ be finite, let $k = \ell + |E^{-1}H|$, and, using Proposition 2.2, choose an incremental sequence $\{\tilde{E}F_{n(j)} g_j\}_{j=1}^\ell$ for $\{\tilde{E}F_n\}$ with $1_G \in \bigcap \tilde{E}F_{n(j)} g_j$.

For $j < i$ we have $g_i \notin \tilde{E}F_{n(j)} g_j$. Since $1_G \in \tilde{E}$ this implies $g_i \notin F_{n(j)} g_j$, hence $\{F_{n(j)} g_j\}_{j=1}^\ell$ is incremental.

For (a), let $i > j$. Then $g_i \notin \tilde{E}F_{n(j)} g_j$ and $E^{-1} \subseteq \tilde{E}$ imply $Eg_i \cap F_{n(j)} g_j = \emptyset$; similarly $g_i \notin \tilde{E}F_{n(j)} g_j$ and $1_G \in F_{n(i)}$, together with the definition of \tilde{E} , give $Eg_i \cap Eg_j = \emptyset$.

For (b), note that since $1_G \in F_n$, by the incremental property, all the g_j are distinct. $Eg_j \cap H \neq \emptyset$ implies $g_j \in E^{-1}H$, so after removing at most $|E^{-1}H|$ elements of the sequence we are left with an incremental sequence of length ℓ which, in addition to the above, satisfies (b).

(c) follows from $1_G \in \tilde{E}F_{n(i)} g_i$. □

Lemma 2.6. *Suppose that $\{F_n\}$ is strongly non-Besicovitch and almost central. Then for every finite $D, E \subseteq G$ there exists a finite $H \subseteq G$, with $D \subseteq H$, satisfying the following property: for every ℓ there is an incremental sequence $\{F_{n(i)} g_i\}_{i=1}^\ell$ such that*

- (i) $Eg_i \cap F_{n(j)} eg_j = \emptyset$ and $Eg_i \cap Eg_j = \emptyset$ for all $i \neq j$ and $e \in E$.

(ii) $Eg_i \cap H = \emptyset$ for all i .

(iii) $F_{n(i)}eg_i \cap H \neq \emptyset$ for every i and $e \in E$.

Proof. Let D, E be given, without loss of generality $1_G \in D \cap E$. Using almost centrality let E' be such that $E'F_n \supseteq F_n(E \cup E^{-1})$, and assume $1_G \in E'$ (otherwise just add 1_G to it). Let $E'' = (E')^{-1}E$, let \widetilde{E}'' be as in the previous lemma, and apply the previous lemma to $H = D(E')^{-1}(\widetilde{E}'')^{-1}$. We obtain arbitrarily long incremental sequences $\{F_{n(i)}g_i\}$ satisfying (a)–(c). Property (ii) is just (b).

For (i), we already know that $E''g_i \cap F_{n(j)}g_j = \emptyset$ for $i > j$. Using the definition of E'' this gives $Eg_i \cap F_{n(i)}Eg_j = \emptyset$ for $i > j$. This is the same as $F_{n(i)}^{-1}Eg_i \cap Eg_j = \emptyset$ for $i > j$, which, by symmetry of $F_{n(i)}$, is just $F_{n(i)}Eg_i \cap Eg_j = \emptyset$ for $i > j$. Thus this relation holds for all $i \neq j$. (i) follows using $1_G \in E \cap F_{n(i)}$.

For (iii), by choice of E' for every $e \in E$ we have $F_{n(i)}e^{-1} \subseteq E'F_{n(i)}$, hence $F_{n(i)} \subseteq E'F_{n(i)}e$. Combined with (c) this implies that $(\widetilde{E}'')^{-1} \cap E'F_{n(i)}eg_i \neq \emptyset$, hence $(E')^{-1}(\widetilde{E}'')^{-1} \cap F_{n(i)}eg_i \neq \emptyset$. (iii) follows since $H \supseteq (E')^{-1}(\widetilde{E}'')^{-1}$. \square

2.2 Necessity

In this section we add the assumption that the sets F_n are symmetric, and continue to assume $1_G \in F_n$. It will be convenient to write $S_F(\varphi, x)$ and $R_F(\varphi, \psi, x)$ instead of $S_F(\varphi)(x)$, $R_F(\varphi, \psi)(x)$.

Theorem 2.7. *If $\{F_n\}$ is strongly non-Besicovitch and almost central then there is an ergodic measure-preserving action of G on a non-atomic measures space (X, \mathcal{B}, μ) , and functions $\varphi, \psi \in L^1(\mu)$ with $\int \psi \neq 0$, such that $R_{F_n}(\varphi, \psi)$ diverges a.e. as $n \rightarrow \infty$.*

Theorem 1.1 is then a formal consequence of Theorem 2.7, since if $\{F_n\}$ is not strongly non-Besicovitch then $\{EF_n\}$ is Besicovitch for some finite set E .

The construction that is the proof of Theorem 2.7 proceeds by cutting and stacking. We give full details below, but let us first give an informal overview for readers familiar with the method. Suppose we have defined a large “stack” whose shape a finite set $E \subseteq G$, and a pair of real-valued functions $\varphi, \psi > 0$ with $\|\varphi\|_1, \|\psi\|_1 < \infty$, corresponding to an \mathbb{R} -coloring of G_0 . Applying the corollary to E and $D = E$ we obtain a set $H \supseteq E$, and, fixing a large N , an incremental sequence $\{F_{m(i)}\gamma_i\}_{i=1}^N$ with the associated properties. Now, cut the original stack into N copies of equal mass, and translate them to $E\gamma_i$, $i = 1, \dots, N$, which by the corollary are pairwise disjoint and disjoint from H . Add new mass to the sites corresponding to H (which is empty) in the new stack, and on it define φ to take very large negative value v , and define ψ to be zero there. Also add new mass where necessary in $\bigcup_{i=1}^N F_{m(i)}E\gamma_i$, defining φ, ψ to be 0 there. If v is negative enough in a manner depending only on the original stack, this forces the

ratios over $F_{n(i)}e\gamma_i$ for $e \in E$ to be ≤ -1 ; but the total change to $\|\varphi\|$ is $|H|v/N$, which can be made arbitrarily small by choosing N large. Iterating this procedure, we can cause the ratios at the points corresponding to the original E to fluctuate between ≥ 1 and ≤ -1 , and in the limit we obtain the desired counterexample.

We now carry this plan out in more detail. First we describe the cutting-and-stacking scheme in the group context. Fix in advance the measure space $(\mathbb{R}, \text{Lebesgue})$. We will define a compatible sequence of partial actions T_n . By this we mean that: (i) for every g we define a sequence of maps T_n^g , $n = 1, 2, \dots$ with increasing domains $X_{n,g} \subseteq \mathbb{R}$ and which extend each other in the sense that $T_{n+1}^g|_{X_{n,g}} = T_n^g$; (ii) for $x \in X$, if both the expressions $T_n^h(T_n^g x)$ and $T_n^{hg}x$ are well defined (that is, if $x \in X_{n,g}$, $T_n^g x \in X_{n,h}$, and $x \in X_{n,hg}$), then they are equal; and (iii) writing $X_n = \bigcup_{g \in G} X_{n,g}$, for every $x \in X_n$ and every $g \in G$ we have $x \in X_{m,g}$ for some $m \geq n$. It is clear that this defines in the limit an action of G on $X = \bigcup_n X_n$ given by $T^g x = \lim_{n \rightarrow \infty} T_n^g x$. At the same time, we will define $\varphi_n, \psi_n : X_n \rightarrow \mathbb{R}$ in a compatible way, giving functions $\varphi, \psi : X \rightarrow \mathbb{R}$ in the limit.

The formulation above is somewhat unwieldy and the construction itself will take the following form. At each stage n we will have defined a finite set $G_n \subseteq G$ and to each $g \in G$ associated an interval $I_{n,g} = [a_{n,g}, b_{n,g}) \subseteq \mathbb{R}$ of length $r_n > 0$, independent of g , and with $I_{n,g} \cap I_{n,h} = \emptyset$ for $g \neq h$. For $x \in I_{n,g}$ the map T_n^h is defined if $hg \in G_n$, in which case $T_n^h x \in I_{n,hg}$ is the point $a_{n,hg} + (a_{n,g} - x)$ that occupies the same position in $I_{n,hg}$ as x occupies in $I_{n,g}$. Thus $X_n = \bigcup_{g \in G_n} I_{n,g}$ and $X_{n,h} = \{x \in X_n : x \in I_{n,g} \text{ and } hg \in G_n\}$. What we have said so far ensures that (ii) is satisfied. To ensure properties (i) and (iii) we first describe the transition from stage n to $n+1$, which is by “cutting and translating”. Given $G_n, \{I_{n,g}\}_{g \in G_n}$ for some n , we first choose a large N and choose elements $\gamma_1, \dots, \gamma_N$ of G such that the sets $G_n \gamma_i$ are pairwise disjoint. Now partition each interval $I_{n,g}$ into N intervals of equal length

$$r_{n+1} = r_n/N$$

Ordering these intervals from left to right, set $I_{n+1,g\gamma_i}$ to be the i -th sub-interval.

We have so far defined intervals for $g \in G'_{n+1} = \bigcup_{i=1}^N G_n \gamma_i$, and one may verify that the compatibility condition (i) holds. To ensure (iii), fix a sequence $\Gamma_n \subseteq G$ of finite subsets increasing to G with $1 \in \Gamma_1$, and define G_{n+1} to be any finite set containing $\Gamma_n G'_{n+1}$; to the new points $g \in G_{n+1} \setminus G'_{n+1}$ assign arbitrary pairwise disjoint intervals $I_{n+1,g} \subseteq \mathbb{R} \setminus X_n$.

We will define by induction $G_n, \{I_{n,g}\}_{g \in G_n}$ as above with associated partial action T_n , and bounded functions $\varphi, \psi : X_n \rightarrow \mathbb{R}$ with $\|\varphi\|_1, \|\psi\|_1 < 2$. Furthermore we will have bounded functions $i_n : X_{n-1} \rightarrow \mathbb{N}$ such that for every $x \in [0, 1]$ the maps $T_n^g x$ are

defined for all $g \in F_{i_n(x)}$, and

$$R_{i_n(x)}(\varphi, \psi, x) = \begin{cases} \geq 1 & n \text{ odd} \\ \leq -1 & n \text{ even} \end{cases}$$

where we define $R_n(\varphi, \psi)$ as before in terms of the partial action T_n . We also will ensure that $i_n(x) \rightarrow \infty$ for $x \in \bigcup_{k=1}^{\infty} X_k$. Assuming all this, it is clear that, for the action T defined in the limit, for every k the ratios $R_n(\varphi, \psi, x)$ diverges for $x \in X_k$, and hence diverge everywhere on $X = \bigcup X_n$. One point we have not touched on is ergodicity of the limit action, we will come back to this below.

It remains to describe the construction. At the first step we set $G_1 = \{1_G\}$, $I_{1,1_G} = [0, 1]$, so $X_1 = [0, 1]$; define φ, ψ and i_1 to be identically 1. Then all the requisite properties hold.

Now suppose for some n we have defined $G_n, \{I_{n,g}\}_{g \in G_n}, r_n, \varphi, \psi$, and i_n as above. For simplicity we assume n is even, the odd case being the same. Let $i_n^* = \sup_{x \in X_{n-1}} i_n(x) < \infty$ and

$$\begin{aligned} \Phi_n &= \sup_{x \in [0,1]} |S_{F_{i_n(x)}}(\varphi, x)| \\ \Psi_n &= \sup_{x \in [0,1]} |S_{F_{i_n(x)}}(\psi, x)| \end{aligned}$$

and choose

$$v = \Phi_n + \Psi_n$$

so that $(v - \Phi_n)/\Psi_n = 1$.

Let H be the set associated to $D = E = G_n$ in Lemma 2.5. Choose N large enough that

$$|H| \cdot v \cdot r_n / N < 2 - \int_{X_n} \varphi$$

Applying the lemma, choose elements $\gamma_1, \dots, \gamma_N$ and indices $k_1, \dots, k_N > i_n^*$ such that

$$H \cap F_{k_j} g \gamma_j \neq \emptyset \quad \text{for all } j \text{ and } g \in G_n \quad (2)$$

$$H \cap G_n \gamma_j = \emptyset \quad \text{for all } j \quad (3)$$

$$G_n \gamma_j \cap F_{k_{j'}} g \gamma_{j'} = \emptyset \quad \text{for } j \neq j' \text{ and } g \in G_n \quad (4)$$

$$G_n \gamma_j \cap G_n \gamma_{j'} = \emptyset \quad \text{for } j \neq j' \quad (5)$$

Let

$$G_{n+1} = \Gamma_{n+1} \left(\left(\bigcup_{j=1}^N F_{k_j} G_{n-1} \gamma_j \right) \cup H \right)$$

Assign intervals of length $r_{n+1} = r_n/N$ to the elements of G_{n+1} as follows: First partition each $I_{n,g}$ into N intervals of length r_{n+1} and for $g \in G_n$ assign to $h = g\gamma_j$ the j -th sub-interval of $I_{n,g}$, which we call $I_{n+1,h}$. So far there are no conflicts by (5) and the assignment consists of disjoint intervals. To the remaining elements $h \in G_{n+1} \setminus \bigcup_{j=1}^N G_n\gamma_j$ associate arbitrary pairwise disjoint intervals $I_{n+1,h} \subseteq \mathbb{R} \setminus X_n$, ensuring that the entire family $\{I_{n+1,g}\}_{g \in G_{n+1}}$ is pairwise disjoint. This can easily be done since $X_n \subseteq \mathbb{R}$ is bounded.

For $x \in X_n \cap I_{n,g\gamma_j}$ for some $g \in G_n$, define $i_{n+1}(x) = k_j$. Again, this is well defined by (5).

On $\bigcup_{h \in H} I_{n+1,h}$ set $\psi \equiv 0$ and $\varphi \equiv v$. There are no conflicts with previous definitions because of (3).

On the remaining mass, define $\varphi \equiv \psi \equiv 0$ for $h \in H$. There are no conflicts by (4).

Finally, in order to verify that $R_{i_{n+1}(x)}(\varphi, \psi, x) \geq 1$ for $x \in X_{n-1}$, note that, by (4), if $x \in I_{n,g\gamma_j}$ for $g \in G_n$ then

$$\begin{aligned} S_{F_{i_{n+1}(x)}}(\varphi, x) &= S_{F_{i_n(x)}}(\varphi, x) + v \cdot |H \cap F_{i_{n+1}(x)}g\gamma_j| \\ S_{F_{i_{n+1}(x)}}(\psi, x) &= S_{F_{i_n(x)}}(\psi, x) \end{aligned}$$

hence, by choice of v and (2), we have $R_{F_{i_{n+1}(x)}}(\varphi, \psi, x) \geq 1$.

While this construction ensures that the ratios diverge on a positive fraction of the mass of a positive fraction of the ergodic component of the action, these ergodic components may, a-priori, be atomic, whereas we require divergence of the ratios on a non-atomic space. The easiest solution is to introduce an intermediate step between the stages of the construction, during which we create a large stack with disjoint but very randomly placed copies of the previous stacks. It is standard to show that the resulting action is ergodic, and the new intermediate steps do not interfere with the construction above. We omit the details.

2.3 Balls in finitely generated groups.

Let G be a finitely generated and B_n balls with respect to some symmetric generating set. These are symmetric and contain the identity and, as noted in the introduction, $\{B_n\}$ are almost central. As noted in Section 2.1, if no infinite subsequence $\{B_{n(i)}\}_{i=1}^\infty$ is Besicovitch then every such subsequence is strongly non-Besicovitch. This and Theorem Theorem 1.3.

The application to the Heisenberg group mentioned after Theorem 1.3 follows from:

Proposition 2.8. *Let*

$$G = \left\{ \left(\begin{array}{ccc} 1 & k & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{array} \right) \mid k, m, n \in \mathbb{Z} \right\}$$

denote the discrete Heisenberg group with generating set

$$\{a^\pm, b^\pm\} = \left\{ \left(\begin{array}{ccc} 1 & \pm 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right), \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & \pm 1 \\ 0 & 0 & 1 \end{array} \right) \right\}$$

Let $\{B_n\}$ be the associated sequence of balls. Then every infinite subsequence $\{B_{n(k)}\}$ is non-Besicovitch.

Proof. Let

$$c = b^{-1}a^{-1}ba = \left(\begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right)$$

and

$$\{c^n\}_{n \in \mathbb{Z}} = \left\{ \left(\begin{array}{ccc} 1 & 0 & m \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \mid m \in \mathbb{Z} \right\}$$

is the center of G . Using the commutation relation $[b, a] = c^{-1}$ it is elementary to show that the set $M_r = \{m : c^m \in B_{4r}\}$ contains gaps that grow arbitrarily large as $r \rightarrow \infty$. Thus we can choose $0 \leq s_r, t_r \in M_r$ such that $(s_r, t_r) \cap M_r = \emptyset$ and $t_r - s_r \rightarrow \infty$. If a sequence $r(i)$ that grows quickly enough (e.g. if $t_{r(i+1)} - s_{r(i+1)} > r(i)^2$), then $\{B_{r(i)}c^{s_{r(i)}-t_{r(i)}}\}$ is an incremental sequence whose elements all contain 1_G . This proves the claim. \square

Theorems 1.3 can be slightly strengthened using the following version of Theorem 2.7:

Theorem 2.9. *In Theorem 2.7, if in addition $F_n = B_{k(n)}$ are balls in a group, then in the conclusion we may also assume that $\varphi, \psi \in L^\infty(\mu)$.*

Proof. In the n -th stage of the construction, instead of setting $\varphi = v$ only on the intervals associated to $h \in H$, choose an appropriate m and set $\varphi = v/|B_{m/3}|$ on the intervals associated to $h \in B_m$. This m is chosen before N and the γ_i , and we can ensure that $G_n \gamma_i \cap B_m = \emptyset$ as in Lemma 2.6 by simply choosing an a-priori larger N and discarding some of its elements, so this does not interfere with the construction.

By choosing N large relative to m , the L^1 -norm of φ still increases arbitrarily little. Now, since $H \cap F_{n(i)}g\gamma_i \neq \emptyset$ for $g \in G_n$ and we can assume that m is large enough that $H \subseteq B_{m/3}$, there is a ball $B_{m/3}\gamma'_i \subseteq B_m \cap F_{k_i}\gamma_i$ for some γ'_i (take γ'_i to be the point on the midpoint geodesic from γ_i to some element of $H \cap F_{n(i)}g\gamma_j$). The proof now carries through. \square

Finally, for finitely generated non-abelian free groups it is elementary that no sequence of balls in the standard generator set is Besicovitch. We omit the proof. We do not know if this is true for every generating set, though it seems very likely that it is.

2.4 Some general reductions

We give here some simple reductions that will be used later. Write $L_+^1(\mu) = \{f \in L^1(\mu) : f \geq 0\}$.

Lemma 2.10. *Let $F_n \subseteq G$. The ratio ergodic theorem holds along $\{F_n\}$ if and only if for every action of G there is a $0 \neq \psi \in L_+^1$, such that $R_{F_n}(\varphi, \psi) \rightarrow \int \varphi / \int \psi$ for all $\varphi \in L_+^1$.*

Proof. One direction is obvious. For the other fix an action and suppose that there is a ψ as above. Convergence of $R_{F_n}(\varphi, \psi)$ for $\varphi \in L_+^1$ implies it for all $\varphi \in L^1$, since the operators $R_{F_n}(\cdot, \psi)$ are linear and one can break an arbitrary L^1 function into a difference of non-negative ones. Now for any $\varphi, \theta \in L^1$ with $\int \theta \neq 0$, the conclusion follows by passing to the limit in the identity $R_{F_n}(\varphi, \theta) = R_{F_n}(\varphi, \psi) / R_{F_n}(\theta, \psi)$. \square

Lemma 2.11. *There exists a sequence $F_n \subseteq G$ along which the ratio ergodic theorem holds if and only if there exists such a sequence with, in addition, $1_G \in F_n$ for all n .*

Proof. Only the “only if” direction must be proved. Suppose the ratio ergodic theorem holds along $\{F_n\}$. First suppose some element $g \in G$ belongs to infinitely many F_n . Let $F_{n(i)}$ be the infinite subsequence of sets containing g and write $F'_i = F_{n(i)}g^{-1}$, so that $1_G \in F'_i$, and ratio ergodic theorem holds along $\{F'_i\}$ because of the identity $R_{F'_i}(\varphi, \psi) = R_{F_{n(i)}}(\varphi, \psi) \circ T^{-g}$.

It remains to deal with the case that no g is in infinitely many F_n . In this case, by passing to a subsequence, we can assume that the sets F_n are pairwise disjoint. We shall show that the ratio ergodic theorem holds along $E_n = \bigcup_{i \leq n} F_i$. Since any $g \in E_1$ belongs to all of the E_n , this brings us back to the first case that was already established.

Thus, consider an action of G on a non-atomic measure space (X, \mathcal{B}, μ) and $\varphi, \psi \in L_+^1(\mu)$. Since $\{F_n\}$ are pairwise disjoint, $S_{E_n}(\psi) = \sum_{k=1}^n S_{F_k}(\psi)$, hence

$$R_{E_n}(\varphi, \psi) = \sum_{k=1}^n \frac{S_{F_k}(\psi)}{S_{E_n}(\psi)} R_{F_k}(\varphi, \psi)$$

Since $R_{F_k}(\varphi, \psi) \rightarrow \int \varphi / \int \psi$, we will be done if we show that $S_{E_n}(\psi) \rightarrow \infty$. To see this choose $A \subseteq X$ and $\varepsilon > 0$ with $\mu(A) > 0$ and $\psi \geq \varepsilon 1_A$. Choose another set $B \subseteq X$ with $\mu(B)/\mu(A)$ irrational. Then $R_{F_n}(1_B, 1_A)$ are rational and converge a.e. to the irrational number $\mu(B)\mu(A)$, so their denominators, which are $S_{F_n}(1_A)$, a.s. tend to ∞ with k . Hence $S_{F_n}(\psi) \geq \varepsilon S_{F_n}(1_A) \rightarrow \infty$, concluding the proof.

We note that if $\bigcup F_n = G$ then $S_{E_n}(\psi) \rightarrow \infty$ follows directly from conservativity. \square

Note that if we only assume the ratios to converge a.e. but not necessarily to the limit $\int \psi / \int \varphi$, then the argument above still works assuming that $S_{E_n}(\psi) \rightarrow \infty$ for $0 \neq \psi \in L_+^1$. This is the case if $\bigcup F_n = G$, for example, because ergodicity on a non-atomic space is the same as conservativity.

We say that $\{F_n\}$ is generating if $\bigcup F_n$ generates G as a group.

Lemma 2.12. *If $\{F_n\}$ does not generate then the ratio ergodic theorem fails along $\{F_n\}$.*

Proof. Suppose F_n lie in a proper subgroup $H < G$ and consider an ergodic action of G whose restriction to H is non-ergodic (e.g. a product measures on $\{0, 1\}^G$ with the shift action). Let \mathcal{I} be the σ -algebra of H -invariant sets and choose functions $\varphi, \psi \in L^1$ that are constant on the atoms of \mathcal{I} but $\mathbb{E}(\varphi|\mathcal{I})/\mathbb{E}(\psi|\mathcal{I})$ is not constant. Clearly $R_{F_n}(\varphi, \psi) = \mathbb{E}(\varphi|\mathcal{I})/\mathbb{E}(\psi|\mathcal{I})$ for all n , so $R_{F_n}(\varphi, \psi) \not\rightarrow \int \varphi / \int \psi$. \square

2.5 The group \mathbb{Z}^∞

We now turn to $G = \mathbb{Z}^\infty$ and the proof of Theorem 1.2, switching to additive notation. The main ingredient is Proposition 2.7 from [6]. In that paper the Besicovitch property is called incompressibility ([6, Definition 1.9]), the two notions are the same by Proposition 2.2.

Proposition 2.13. *If $F_n \subseteq \mathbb{Z}^\infty$ are finite sets and $\{F_n\}$ generates, then $\{F_n\}$ is not Besicovitch.*

Combined with the fact that any sequence in an abelian group is almost central, the proposition above and Theorem 2.7 immediately implies that the ratio ergodic

theorem fails along every generating symmetric sequence $F_n \subseteq \mathbb{Z}^\infty$ with $0_G \in F_n$. We now show that the symmetry assumption is not necessary:

Proposition 2.14. *The ratio ergodic theorem fails along any generating sequence $F_n \subseteq \mathbb{Z}^\infty$ with $0_G \in F_n$.*

Proof. Suppose that $\{F_n\}$ satisfies the ratio ergodic theorem and $1_G \in F_n$. Then the same is true for $\{-F_n\}$. To see this, given an action $\{T^g\}_{g \in \mathbb{Z}^\infty}$ define an action $\tilde{T}^g = T^{-g}$ (this is an action because \mathbb{Z}^∞ is abelian). Then $\sum_{g \in -F_n} \varphi(T^g x) = \sum_{g \in F_n} \varphi(\tilde{T}^g x)$, and so the ratios over $-F_n$ with respect to T are the same as the ratios over F_n with respect to T , and so converge as required.

Now let $E_n = F_n \cup (-F_n)$, which is a symmetric sequence with $0_G \in E_n$. Define a probability measures ν_n on E_n by

$$\nu_n = \frac{1}{2|F_n|} \sum_{g \in \pm F_n} \delta_g$$

For a finitely supported probability measure ν on G , let

$$S_\nu(\varphi) = \int \varphi(T^g x) d\nu(g)$$

and define $R_\nu(\varphi, \psi) = S_\nu(\varphi)/S_\nu(\psi)$. Then for any action and φ, ψ as in the ratio ergodic theorem,

$$R_{\nu_n}(\varphi, \psi) = \frac{1}{2}(R_{F_n}(\varphi, \psi) + R_{-F_n}(\varphi, \psi)) \rightarrow \int \varphi / \int \psi \quad \mu\text{-a.s.}$$

While $R_{\nu_n} \neq R_{E_n}$, on non-negative functions the two differ by at most a multiplicative constant of 4, since ν_n is equivalent to the uniform measure u_n on E_n with Radon-Nikodym derivative between 1 and 2, and $R_{E_n} = R_{u_n}$. Thus, if there is an action and functions $\varphi, \psi \in L_+^1$, $\int \psi \neq 0$, such that $R_{E_n}(\varphi, \psi)$ fluctuates wildly enough (e.g. $\limsup / \liminf > 4$), then we have a contradiction to the convergence of R_{ν_n} . Since $\{E_n\}$ is symmetric, contains 1_G , and is almost central and strongly non-Besicovitch (see discussion preceding this proposition), such an action and pair of functions can be constructed using exactly the same scheme as in the proof of Theorem 2.7. We omit the details. \square

Now, we have already seen that if there is some sequence along which the ratio ergodic theorem holds then there is also such a sequence that contains 0_G (Lemma 2.11) and generates (2.12). With these facts in hand, the proposition above proves Theorem 1.2.

3 Groups of polynomial growth

In this section we prove the ratio ergodic theorem “in density” for groups of polynomial growth (Theorem 1.4). After defining the sequence $F_n \subseteq G$ in Section 3.1, the proof follows the standard two-step scheme: in Section 3.2 we prove, for fixed ψ , that $R_{F_n}(\varphi, \psi)$ converges to the proper limit on a dense family of functions $\varphi \in L^1$ (a “Chacon-Ornstein lemma”), and in Section 3.3 we extend to all $\varphi \in L^1$ using a suitable maximal inequality. Both parts use growth properties of G in an essential way.

3.1 The averaging sequence

Let G be a group of polynomial growth and $B_n = A^n$ the balls with respect to some symmetric generating set A . By Gromov’s theorem [5], G is virtually nilpotent, and a theorem of Bass [1] implies that there are constants c_1, c_2, c (moreover, with $c \in \mathbb{N}$) such that

$$c_1 n^c \leq |B_n| \leq c_2 n^c$$

Define the k -boundary of B_n to be

$$\partial_k B_n = B_{n+k} \setminus B_{n-k}$$

We remark that it is easy to show that $\{B_n\}$ is a Følner sequence, but we will not use this fact.

We now define the subsequence $F_i = B_{n(i)}$ for which we will prove Theorem 1.4 (the construction below can be perturbed in many ways to get a large class other such sequences). Let

$$J_m = [2^{m-1}, 2^m) \cap \mathbb{Z}$$

We define the index sequence $n(i)$ for $i \in J_m$, recursively in $m = 1, 2, 3, \dots$. For $m = 1$ set $n(1) = 1$. Now assume we have defined n_i for $i \in \bigcup_{k < m} J_k$. Let

$$N(m) = n(2^{m-1} - 1)$$

which is the largest value of $n(i)$ defined so far, and set $\{n(i)\}_{i \in J_m}$ to be the arithmetic sequence with $|J_m|$ terms and gap $3N(m)$, starting at

$$L(m) = |J_m| \cdot 3N(m)$$

Thus $n(2^{m-1} + i) = L(m) + i \cdot 3N(m)$ for $0 \leq i < 2^m$.

Note that $\{n(i)\}_{i \in J_m} \subseteq [L(m), 2L(m))$, hence $N(m) \leq 2L(m-1)$, and, since by

the last equation $L(m-1) = 2^{m-1} \cdot 3N(m-1)$, we deduce that

$$N(m) \leq (62^{m-1})^m$$

Having defined $F_i = B_{n(i)}$, for $i \in J_m$, set

$$F_i^+ = B_{n(i)+N(m)}$$

and

$$\partial^* F_i = \partial_{N(m)} F_i$$

Notice that $F_{i-1}^+ \cup \partial^* F_i \subseteq F_i^+$ and $\partial^* F_i \cap F_{i-1}^+ = \emptyset$.

3.2 Convergence on a dense subset of $L^1(\mu)$

For the rest of the section, fix an ergodic measure-preserving action of G on a σ -finite measure space (X, \mathcal{B}, μ) . Given $\varphi : X \rightarrow [0, \infty)$ write

$$\varphi_i(x) = \frac{\sum_{g \in \partial^* F_i} \varphi(T^g x)}{\sum_{g \in F_{i-1}^+} \varphi(T^g x)} \quad (6)$$

Lemma 3.1. *Let φ be as above and $x \in X$. Given $\varepsilon, \delta > 0$ suppose that $N \in J_m$ and $U \subseteq \{1, \dots, N\}$ are such that $|U|/N \geq \delta$ and $\varphi_i(x) > \varepsilon$ for $i \in U$. Then, assuming m is large enough in a manner depending only on ε, δ ,*

$$S_{F_N^+}(\varphi, x) \geq |F_N^+|^2 \varphi(x)$$

Proof. We suppress x in our notation. For $i \in U$ we have by definition that $S_{\partial^* F_i}(\varphi) \geq \varepsilon S_{F_{i-1}^+}(\varphi)$ and hence $S_{F_i^+}(\varphi) \geq (1 + \varepsilon) S_{F_{i-1}^+}(\varphi)$. Since $\varphi \geq 0$, for any $j < i$ we have $S_{F_i^+}(\varphi) \geq S_{F_j^+}(\varphi)$. Starting from $S_{F_N^+}(\varphi)$ and applying this recursively to the elements of U in reverse order, we have

$$\begin{aligned} S_{F_N^+}(\varphi) &\geq (1 + \varepsilon)^{|U|-1} S_{F_1^+}(\varphi) \\ &\geq (1 + \varepsilon)^{\delta N - 1} \varphi \\ &\geq (1 + \varepsilon)^{\delta 2^{m-1}} \varphi \end{aligned}$$

It remains to notice that

$$|F_N^+| \leq |B_{2n(N)}| \leq Cn(N)^c \leq CN(m+1)^c \leq C(2^m)^{cm} = C2^{cm^2}$$

for some constant C depending only on c_2 , and that if m is large in a manner depending

on ε, δ then $C2^{2cm^2} \leq (1 + \varepsilon)^{\delta 2^{m-1}}$. \square

Theorem 3.2. *Let $\varphi \in L^1(\mu)$ with $\varphi \neq 0$ and $\varphi \geq 0$. Then $\varphi_i \xrightarrow{\bar{d}} 0$ ν -a.e., where $d\nu = \varphi d\mu$.*

Proof. Fix $\varepsilon > 0$. It suffices to show that

$$\bar{d}(i : \varphi_i(x) > \varepsilon) = 0 \quad \nu\text{-a.e.}$$

Fix $\delta > 0$, which we suppress in our notation, and let

$$E_N = \left\{ x \in X \left| \begin{array}{l} \delta < \varphi(x) < \delta^{-1} \text{ and } \varphi_i(x) > \varepsilon \text{ for} \\ \text{at least a } \delta\text{-fraction of } 1 \leq i \leq N \end{array} \right. \right\}$$

It is enough to show, for every $\delta > 0$, that ν -a.e. x belongs to only finitely many E_N .

We establish the last claim. Assume, as we may, that m is large relative to ε, δ as in the previous lemma. Let $N \in J_m$. By invariance of μ , we have

$$\begin{aligned} |F_N| \cdot \nu(E_N) &= \int \sum_{g \in F_N} 1_{E_N}(T^g x) \varphi(T^g x) d\mu(x) \\ &= \int S_{F_N}(\varphi \cdot 1_{E_N}) d\mu(x) \end{aligned} \tag{7}$$

Suppose that $g \in F_N$ is such that $T^g x \in E_N$. By the previous lemma (applied to $\varphi \circ T^g$) and the definition of E_N ,

$$\begin{aligned} S_{F_N^+}(\varphi)(T^g x) &\geq |F_N|^2 \cdot \varphi(T^g x) \\ &\geq |F_N|^2 \cdot \delta \end{aligned}$$

Since $hg \in (F_N^+)^2$ for every $h \in F_N^+$, we have shown that if $N \in J_m$ then

$$S_{F_N}(\varphi \cdot 1_{E_N}) > 0 \quad \implies \quad S_{(F_N^+)^2}(\varphi) > \delta |F_N|^2$$

By definition of E_N we have $\varphi(T^g y) < \delta^{-1}$ if $1_{E_N}(T^g y) \neq 0$. Therefore $S_{F_N}(\varphi \cdot 1_{E_N}) \leq |F_N| \cdot \delta^{-1}$ so, by the implication above,

$$S_{F_N}(\varphi \cdot 1_{E_N}) \leq |F_N| \cdot \delta^{-1} \cdot 1_{\{S_{(F_N^+)^2}(\varphi) > \delta |F_N|^2\}}$$

Integrating this $d\mu$ and using (7) and Markov's inequality,

$$\begin{aligned}
|F_N|\nu(E_N) &\leq |F_N| \cdot \delta^{-1} \cdot \mu\left(x : S_{(F_N^+)^2}(\varphi) > \delta|F_N|^2\right) \\
&\leq |F_N| \cdot \delta^{-2} \left(|F_N|^{-2} \cdot \int S_{(F_N^+)^2}(\varphi) d\mu\right) \\
&= \delta^{-2} \cdot |F_N|^{-2} \cdot |(F_N^+)^2| \cdot \int \varphi d\mu
\end{aligned}$$

Now by polynomial growth and the fact that $(F_N^+)^2 \subseteq B_{4n(N)}$ we have

$$|(F_N^+)^2| \leq C \cdot |F_N|$$

for a constant C depending on c_1, c_2, c , but not on m . Thus, we have shown

$$\nu(E_N) \leq \frac{C \int \varphi d\mu}{\delta^2 |F_N|^2} \leq \frac{C \int \varphi d\mu}{\delta^2 N^2}$$

using the trivial bound $|F_N| \geq N$. This is summable, so by Borel-Cantelli, ν -a.e. x belongs to finitely many E_N . \square

Recall that a co-boundary is a function of the form $\varphi = \tau - \tau^g$ for some $g \in G$. It is said to be an L^1 -co-boundary if $\tau \in L^1(\mu)$, and positive if $\tau \geq 0$. As a consequence of the theorem above we obtain a Chacon-Ornstein type statement. Note that in what follows, σ -finiteness ensures that statements about strictly positive L^1 -functions are not vacuous.

Corollary 3.3. *Let $\tau \in L^1(\mu)$ with $\tau > 0$ and $\tau \neq 0$. Then for every $g \in G$,*

$$\bar{d}\text{-}\lim_{i \rightarrow \infty} R_{F_i}(\tau - \tau^g, \tau) = 0 \quad \mu\text{-a.e.}$$

Proof. There is an i_0 such that $g \in F_{i_0}$, and for $i > i_0$ we have

$$|R_{F_i}(\tau - \tau^g, \tau)| \leq \left| \frac{S_{\partial^* F_i}(\tau)}{S_{F_i}(\tau)} \right| \leq \left| \frac{S_{\partial^* F_i}(\tau)}{S_{F_{i-1}^+}(\tau)} \right| = \tau_i$$

where τ_i is defined as in (6), and we have used $\tau \geq 0$. From the theorem we conclude that $R_{F_i}(\tau - \tau^g, \tau) \xrightarrow{\bar{d}} 0$ at $\tau d\mu$ -a.e. point. Since $\tau > 0$ the measures μ and $\tau d\mu$ are equivalent, and the corollary follows. \square

The next conclusion is standard from the previous one.

Proposition 3.4. *Given $0 < \psi \in L^1(\mu)$, the set of $\varphi \in L^1(\mu)$ such that $R_{F_i}(\varphi, \psi) \xrightarrow{\bar{d}} \int \varphi / \int \psi$ a.e. is dense in L^1 .*

Proof. Let us say that $\tau \in L^1$ is ψ -dominated if $0 < \tau < M\psi$ for some $M = M(\tau)$. We claim, first that the convergence in the statement holds for $\varphi = \tau - \tau^g$ where $g \in G$ and $\tau \in L^1$ is ψ -dominated; and second, that the joint linear span of ψ and the set of such φ is dense in L^1 . The two claims prove the proposition since the limit in question holds trivially when $\varphi = \psi$ and the operators $R_{F_i}(\cdot, \psi)$ are linear.

For the first statement, let $\varphi = \tau - \tau^g$ with $0 < \tau \leq M\psi$. Then $|R_{F_n}(\tau, \psi)| \leq M$, hence by the previous corollary,

$$R_{F_i}(\tau - \tau^g, \psi) = R_{F_i}(\tau - \tau^g, \tau) \cdot R_{F_i}(\tau, \psi) \xrightarrow{\bar{d}} 0$$

For the second statement, observe that since $\psi > 0$, the set of differences of ψ -dominated functions is dense in the positive cone of L^1 . It follows easily that the linear span of the set of co boundaries $\varphi = \tau - \tau^g$ with τ a ψ -dominated function is dense among all L^1 -co-boundaries (note that in general a co-boundary splits into the difference of two positive co-boundaries). We now refer to the standard fact that, for ergodic actions and assuming $\int \psi \neq 0$, the linear span of ψ and the L^1 -co-boundaries is dense subspace of $L^1(\mu)$ (see e.g. [4]). \square

3.3 A density version of the maximal inequality

The next step is to prove a maximal-type inequality that will allow to go from the \bar{d} -convergence of $R_{F_i}(\varphi, \psi)$ on a dense set of $\varphi \in L^1(\mu)$ to all of $L^1(\mu)$. Define the density-limsup by

$$\bar{d}\text{-}\limsup_{n \rightarrow \infty} a_n = \inf\{t \in \mathbb{R} : \bar{d}(n : a_n > t) = 0\}$$

Lemma 3.5. *If $a_n : X \rightarrow \mathbb{R}$ are measurable then $a = \bar{d}\text{-}\limsup a_n$ is measurable.*

Proof. It suffices to show that $\delta_t(x) = \bar{d}(n : a_n(x) > t)$ is measurable for each fixed t , and this is obvious since $\delta_t(x) = \limsup \frac{1}{N} \sum_{n=1}^N 1_{\{a_n > t\}}(x)$. \square

Theorem 3.6. *Let $\varphi, \psi \in L^1(\mu)$ with $\varphi, \psi \geq 0$ and $\int \psi d\mu \neq 0$, and write $d\nu = \psi d\mu$. Then*

$$\nu \left(\bar{d}\text{-}\limsup_{n \rightarrow \infty} R_{F_n}(\varphi, \psi) > t \right) \leq C \frac{\int \varphi d\mu}{t}$$

Before giving the proof of the maximal inequality, let us use it to complete the proof of the ratio ergodic theorem. Fix $0 < \psi \in L^1(\mu)$; by Lemma 2.10 it suffices to prove $R_{F_n}(\varphi, \psi) \rightarrow \int \varphi / \int \psi$ for $\varphi \in L^1(\mu)$. Noting that $R_{F_n}(\varphi - c\psi, \psi) = R_{F_n}(\varphi, \psi) - c$ and setting $c = \int \varphi / \int \psi$, we may further assume that $\int \varphi d\mu = 0$. Let $\varepsilon > 0$ and let $\varphi' \in L^1(\mu)$ be such that $R_{F_n}(\varphi', \psi) \xrightarrow{\bar{d}} 0$ and $\|\varphi - \varphi'\|_1 < \varepsilon$, as exists by the previous

proposition. By Theorem 3.6,

$$\nu \left(\bar{\text{d-limsup}}_{n \rightarrow \infty} R_{F_n}(|\varphi - \varphi'|, \psi) > \frac{1}{2}\sqrt{\varepsilon} \right) < C \frac{\|\varphi - \varphi'\|_1}{\sqrt{\varepsilon}} < C\sqrt{\varepsilon}$$

and by the triangle inequality and $R_{F_n}(\varphi', \psi) \xrightarrow{\bar{d}} 0$,

$$\nu \left(\bar{\text{d-limsup}}_{n \rightarrow \infty} |R_{F_n}(\varphi, \psi)| > \sqrt{\varepsilon} \right) \leq \nu \left(\bar{\text{d-limsup}}_{n \rightarrow \infty} |R_{F_n}(|\varphi - \varphi'|, \psi)| > \sqrt{\varepsilon} \right) < C\sqrt{\varepsilon}$$

From this we conclude that

$$\nu \left(\bar{\text{d-limsup}}_{n \rightarrow \infty} |R_{F_n}(\varphi, \psi)| > 0 \right) = 0$$

Since $\psi > 0$ the measures ν and μ are equivalent, so this is the same as $R_{F_n}(\varphi, \psi) \xrightarrow{\bar{d}} 0$ μ -a.e., as desired.

Turning to the maximal inequality, we will use the following Besicovitch-type property:

Lemma 3.7. *There is a constant C such that for any k, N , if $\{B_{r(i)}g_i\}_{i=1}^k$ is an incremental sequence such that $r(i) \in [N, 2N]$, then its multiplicity is at most C .*

Proof. From the assumption $B_{[r(i)/2]}g_i$ are pairwise disjoint sets of size $\geq c_1(N/2)^c$. If $h \in \bigcap_{i \in I} B_{r(i)}g_i$ for some $I \subseteq \{1, \dots, k\}$ then $B_{[r(i)/2]}g_i \subseteq B_{3N}h$ for $i \in I$, and the maximal number of such balls is therefore $|B_{3N}h|/|B_{[N/2]}|$. Since $|B_{3N}| \leq c_2(3N)^c$ we have $|I| \leq 6^c c_2/c_1$. \square

Next we apply a variant of the Vitali covering argument.

Lemma 3.8. *Let $\alpha > 0$, let $\varphi, \psi : X \rightarrow [0, \infty)$ and $x \in X$, let N be given and $E \subseteq F_N$. Suppose that for each $g \in E$ there is an $1 \leq i(g) \leq N$ such that $\varphi_{j(g)}(T^g x) < \varepsilon$ and $\psi_{j(g)}(T^g x) < \varepsilon$, where φ_i, ψ_i are defined as in (6). Also suppose that $R_{j(g)}(\varphi, \psi, T^g x) > \alpha$. Then*

$$\sum_{g \in E} \psi(T^g x) \leq \frac{C}{(1 - \varepsilon)\alpha} S_{F_N^2}(\varphi, x)$$

Proof. Let $E_m = \{g \in E : n(g) \in J_m\}$. Let M be the maximal value of m for which $E_m \neq \emptyset$. Define $E'_m \subseteq E_m$ recursively starting from $m = M$ and working down to $n = 1$: assuming we have defined E'_k for $k > m$, define $E'_m = \{g_{m,1}, \dots, g_{m,\ell(m)}\}$ to be a maximal sequence satisfying the property in the hypothesis of the previous lemma with respect to $r(g) = n(i(g))$, and also satisfying $g_i \notin \bigcup_{k>m} \bigcup_{i=1}^{\ell(k)} B_{r(g_{k,i})}g_{k,i}$.

It is easily seen by induction that

$$E_m \subseteq \bigcup_{k \geq m} \bigcup_{i=1}^{\ell(k)} F_{j(g_{k,i})} g_{k,i}$$

For $h \in E_m$ let

$$F'_{j(h)} h = F_{j(h)} h \setminus \bigcup_{k < m} \bigcup_{i=1}^{\ell(k)} F_{j(g_{k,i})} g_{k,i}$$

so that $\bigcup_{h \in E} F'_{j(h)} h = \bigcup_{h \in E} F_{j(h)} h$. Given $g \in G$ and m , by the previous lemma g belongs to at most C of the sets $F_{j(g_{m,i})} g_{m,i}$, therefore and if $m_0 = m_0(g)$ is the least index such that this is true for some j and $g \in F_{j(g_{m,i})} g_{m,i}$ then g belongs to $F'_{j(h)} h$ only for some $h \in E_{m_0}$ (but no elements $h \in E_{m'}$ for $m' \neq m_0$), and to at most C such sets. It follows that

$$\sum_{g \in E} \psi(T^g x) \leq C \sum_m \sum_{i=1}^{\ell(m)} \sum_{g \in F'_{j(g_{m,i})} g_{m,i}} \psi(T^g x)$$

Now by our assumptions about $\varphi_i(T^g x)$ and $\psi_i(T^g x)$ for $g \in E$, and the fact that $F_{j(h)} h \setminus F'_{j(h)} h \subseteq \partial^* F_{j(h)} h$, we conclude that for all m and $1 \leq i \leq \ell(m)$,

$$\begin{aligned} \sum_{g \in F'_{j(g_{m,i})} g_{m,i}} \psi(T^g x) &\leq \sum_{g \in F_{j(g_{m,i})} g_{m,i}} \psi(T^g x) \\ &\leq \alpha^{-1} \sum_{g \in F_{j(g_{m,i})} g_{m,i}} \varphi(T^g x) \\ &\leq \frac{\alpha^{-1}}{1 - \varepsilon} \sum_{g \in F'_{j(g_{m,i})} g_{m,i}} \varphi(T^g x) \end{aligned}$$

Combined with the previous inequality, this gives

$$\begin{aligned} S_{F_N}(\psi 1_E, x) &\leq \frac{C\alpha^{-1}}{1 - \varepsilon} \sum_m \sum_{i=1}^{\ell(m)} \sum_{g \in F'_{j(g_{m,i})} g_{m,i}} \varphi(T^g x) \\ &\leq \frac{C\alpha^{-1}}{1 - \varepsilon} \sum_{g \in F_N E} \varphi(T^g x) \\ &\leq \frac{C\alpha^{-1}}{1 - \varepsilon} S_{F_N^2}(\varphi, x) \end{aligned}$$

because $\varphi \geq 0$ and $F_N E \subseteq F_N^2$; the claim follows. \square

Proof of the maximal inequality (Theorem 3.6). Since it suffices to prove the claim with φ replaced by $\varphi + \rho\psi$ for arbitrarily small ρ , we can assume that $\varphi > 0$. Write $R = \bar{d}\text{-}\limsup R_{F_i}(\varphi, \psi)$. Fix $t > 0$ and denote

$$S = \{x : R(x) > t\}$$

For $\delta > 0$ let

$$S_\delta = \{x : \bar{d}(R_{F_i}(\varphi, \psi, x) > t + \delta) > \delta\}$$

Since $S = \bigcup_{\delta>0} S_\delta$ and the union is monotone, it suffices for us to show that $\nu(S_\delta) \leq \frac{C}{1-\delta} \int \varphi/t$ for a constant C independent of δ . By Theorem 3.2, for $\varphi d\mu$ -a.e. $x \in S_\delta$ we have $\varphi_i(x) \xrightarrow{\bar{d}} 0$, with φ_i as in (6). Since $\varphi, \psi > 0$ the measures $\varphi d\mu$ and $\nu = \psi d\mu$ are equivalent, so this is also true ν -a.e., hence the set

$$S'_\delta = \{x : \bar{d}(R_{F_i}(\varphi, \psi, x) > t + \delta \text{ and } \varphi_i(x) < \delta) > \delta\}$$

differs from S_δ on a set of ν -measure 0, and it suffices to bound $\nu(S'_\delta)$. Now, since ν is a finite measure, there is an N such that

$$S'_{\delta,N} = \{x : R_{F_i}(\varphi, \psi, x) > t + \delta \text{ and } \varphi_i(x) < \delta \text{ for some } 1 \leq i \leq N\}$$

satisfies

$$\nu(S'_{\delta,N}) > \frac{1}{2} \nu(S'_\delta)$$

and so it suffices to bound the measure of $S'_{\delta,N}$.

This now is a direct application of the transference principle and the previous lemma. We have

$$|F_N| \nu(S'_{\delta,N}) = \int S_{F_N}(\psi 1_{S'_{\delta,N}}) d\mu$$

For $x \in X$ let

$$E = E_x = \{g \in F_N : T^g x \in S'_{\delta,N}\}$$

and for $g \in E$ define $i(g) = i_x(g) \in \{1, \dots, N\}$ to be an index such that $R_{F_{j(g)}}(\varphi, \psi, T^g x) > t$ and $\varphi_{i(g)}(T^g x) < \delta$. In this notation,

$$S_{F_N}(\psi 1_{S'_{\delta,N}}, x) = \sum_{g \in E_x} \psi(T^g x)$$

and we may apply Lemma 3.8 at each x , concluding that

$$|F_N| \nu(S'_{\delta,N}) \leq \frac{C}{(1-\delta)t} \int S_{F_N^2}(\varphi) d\mu = \frac{C}{(1-\delta)} |F_N^2| \int \varphi d\mu$$

The conclusion now follows from the fact that by polynomial growth, $|F_N^2|/|F_N| = |B_{2n(N)}|/|B_{n(N)}|$ is bounded uniformly in N . \square

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